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Translated by B. D.

## ON A CERTAIN NONLINEAR EQUATION OF ELECTROHYDRODYNAMICS AND MAGNETOELASTICITY

PMM Vol. 35, №6, 1971, pp. 1038 - 1046
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(Received January 1, 1971)

Properties of the solutions of the Emden-Fowler equation the nonlinear term of which contains the unknown function raised to a negative power, are determined. Boundary value problems in which one of the conditions corresponds to the requirement that the solution be bounded when the argument is equal to zero or infinity (this requirement occurs in a number of problems in mechanics) are also considered. These boundary value problems may have any number, or even an enumerable set of solutions, the latter case characterized by the dependence of these solutions on a parameter of an unusual form.

For $n > 0$ ($n$ is the power in which the unknown function appears in the nonlinear term) the Emden-Fowler equation has been studied exhaustively in [1, 2]. The problems of electrohydrodynamics and nonlinear magnetoelasticity leading to the Emden-Fowler equation with $n = -2$, were studied in [3 - 5]. The present paper deals with yet another problem, namely that of equilibrium of heavy filaments through which a current flows. This problem leads to the case of $n = -1$ and the nonlinear term may be positive or negative, depending on whether the filaments attract or repel each other.

**1. Reduction to an autonomous system and properties of its solutions.** The Emden-Fowler equation has the form [1]

$$\frac{d}{d\rho}\left(\rho^{\alpha}\frac{dw}{d\rho}\right) \pm \rho^{\sigma} w^{n} = 0 \tag{1.1}$$

We shall consider the values $w > 0$. When $n$ is rational and its denominator is odd (in this case real solutions $w < 0$ exist), the substitution $w_1 = -w$ reduces the case $w < 0$ to one of the cases given below. We shall also assume that $\rho > 0$. Introducing a new argument $\tau$ and the unknown $\eta$ by means of the relations $\rho = ae^{-\beta\tau}$ and $w = b\eta e^{-\delta\tau}$ we arrive at the following equation

$$\eta'' - (2\delta - \beta + \alpha\beta)\eta' + \delta(\delta - \beta + \alpha\beta)\eta \pm \tag{1.2}$$

$$\pm a^{2+\sigma-\alpha} b^{n-1}\beta^2\eta^n \exp\{[\delta(1-n) - \beta(2+\sigma-\alpha)]\tau\} = 0 \quad (\eta' = d\eta/d\tau)$$

Let us now assume that

$$a^{2+\sigma-\alpha}b^{n-1}\beta^2 = 1, \qquad \delta = 1, \qquad (2+\sigma-\alpha)\beta = 1-n$$

(the case $2+\sigma-\alpha = 0$ will be considered separately). Then $\tau$ will not appear in (1.2) explicitly (this can also be achieved by means of a different substitution [4, 6], and (1.2) can be replaced by

$$\eta' = \vartheta, \quad \vartheta' = (c+1)\vartheta - c\eta + \gamma\eta^n, \quad \gamma = \pm 1, \quad c = 1 - \beta + \alpha\beta \tag{1.3}$$

From (1.3) the following relations to be utilised later are:

$$(c-1)\eta(\tau) = (c\eta_0 - \vartheta_0)e^{\tau} - (\eta_0 - \vartheta_0)e^{c\tau} + \gamma\int_0^{\tau}[e^{c(\tau-\xi)} - e^{\tau-\xi}]\eta^n(\xi)\,d\xi$$

$$(c-1)\vartheta(\tau) = (c\eta_0 - \vartheta_0)e^{\tau} - c(\eta_0 - \vartheta_0)e^{c\tau} + \gamma\int_0^{\tau}[ce^{c(\tau-\xi)} - e^{\tau-\xi}]\eta^n(\xi)\,d\xi$$

$$(\eta_0 = \eta(0), \vartheta_0 = \vartheta(0)) \tag{1.4}$$

Let us investigate various combinations of the signs of $c$ and $\gamma$.

1. Let $c > 0$ and $\gamma = 1$. Then the system (1.3) has a singularity $\eta = \eta_c$, $\vartheta = 0$, $\eta_c^{n-1} = c$. Setting $\eta - \eta_c = \zeta$ and linearizing (1.3) near the singularity we obtain

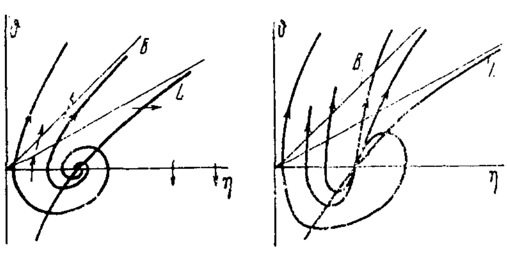
$$\zeta' = \vartheta, \quad \vartheta' = (c+1)\vartheta - c(1-n)\zeta \tag{1.5}$$

The coefficients of (1.5) determine the character of the singularity.

Let us consider a subcase, when $(c-1)^2 + 4nc < 0$ and the singularity is an unstable focus. Making use of the signs of the derivative

$$d\vartheta/d\eta = \vartheta^{-1}[(c+1)\vartheta - c\eta + \gamma\eta^n] \tag{1.6}$$

we shall mark the directions of the integral curves $\vartheta(\eta)$ on the half-plane $O\eta\vartheta$, $\eta > 0$ (these are shown in Fig. 1a and their directions correspond to increasing $\tau$) We first assume that $c > 1$. When $(c+1)\vartheta - c\eta + \gamma\eta^n = 0$ on $L$ we have $d\vartheta/d\eta = 0$, while on $B$ we have $d\vartheta/d\eta = 1 + \eta^{n-1}\vartheta^{-1} > 1$ for $\vartheta = \eta$. Taking also into account the signs $d^2\vartheta/d\eta^2$ we find that the $O\vartheta$-axis cannot be an asymptote to $\vartheta(\eta)$ (below we shall show that this is possible in other cases) and, that the line $L$ leaves, after intersecting $B$, into infinity while the line intersecting $L$ subsequently intersects $O\eta$.




(a) (b)

Fig. 1.

When $\eta > \eta_c$ the space between $B$ and $O\eta$ contains only those lines $\vartheta(\eta)$ which either go to infinity, or intersect $L$ and return. From this it follows that an integral curve going to infinity below $B$ (a dividing curve) exists. This dividing curve is unique and $B$ is its asymptote.

Thus when $n \leqslant -1$ and $\tau$ increases, all integral curves except the dividing curve and the singular curve $\eta \equiv \eta_c$ rotate about the focus an infinite number of times, intersect $B$ and go to infinity, while the dividing curve, having left the focus, approaches $B$ from below asymptotically (Fig. 1a). When $n > -1$ there are curves arriving from the focus to $O\vartheta$ and curves leaving $O\vartheta$ and going to infinity.

The most interesting problem of the theory of the Emden-Fowler equation is that of the behavior of its solutions when $\rho \to 0$ and $\rho \to \infty$. For the system (1.3) this corresponds to the behavior of its solutions when $\tau \to \pm\infty$. The case $\tau \to -\infty$ is obvious. In the case of $\tau \to \infty$ the integral curves go to infinity, the nonlinear term in (1.3) decreases and the behavior of the solutions is basically determined by the linear part of the equation.

Let us take any curve which is neither the dividing curve, nor the singular curve $\eta = \eta_c$. Multiplying both sides of (1.4) by $e^{-c\tau}$ and passing to the limit we find that finite limits

$$\lim_{\tau\to\infty} \eta(\tau) e^{-c\tau} = (c-1)^{-1}\Big[\vartheta_0 - \eta_0 + \int_0^\infty e^{-c\xi}\eta^n(\xi)\, d\xi\Big] > 0$$

$$\lim_{\tau\to\infty} \vartheta(\tau) e^{-c\tau} = c \lim_{\tau\to\infty} \eta(\tau) e^{-c\tau} \qquad (1.7)$$

exist. Let us consider the dividing curve, utilizing the identity

$$\eta(\tau) = \eta_0 e^{\tau} + \int_0^\tau e^{\tau-\xi}[\vartheta(\xi) - \eta(\xi)]\, d\xi \qquad (1.8)$$

Since $\eta(\tau) - \vartheta(\tau) \to 0$ as $\tau \to \infty$, the integral

$$\int_0^\infty e^{-\xi}[\vartheta(\xi) - \eta(\xi)]\, d\xi$$

converges and the following finite limits exist

$$\lim_{\tau\to\infty} \eta(\tau) e^{-\tau} = \eta_0 - \int_0^\infty e^{-\xi}[\eta(\xi) - \vartheta(\xi)]\, d\xi > 0$$

$$\lim_{\tau\to\infty} \vartheta(\tau) e^{-\tau} = \lim_{\tau\to\infty} \eta(\tau) e^{-\tau} \qquad (1.9)$$

To investigate the boundary value problems (see below) we must know how the above limits are affected by the change of the initial point on the trajectory. Consider the limits of the functions $f(\tau)\eta_1(\tau)$ and $f(\tau)\eta_2(\tau)$, where $\eta_1(\tau)$ and $\eta_2(\tau)$ correspond to the same integral curve, but have different values when $\tau = 0$. Let $\eta_1(0) = \eta_{10}$, $\vartheta_1(0) = \vartheta_{10}$, $\eta_2(0) = \eta_{20}$, and $\vartheta_2(0) = \vartheta_{20}$. Then $\eta_2(\tau) = \eta_1(\tau + \tau_{12})$, where $\tau_{12}$ is such that $\eta_1(\tau_{12}) = \eta_{20}$ and $\vartheta_1(\tau_{12}) = \vartheta_{20}$; $\tau_{12} > 0$, if the passage from $(\eta_{10}, \vartheta_{10})$ to $(\eta_{20}, \vartheta_{20})$ takes place in the direction of increasing $\tau$ and $\tau_{12} < 0$ if the passage corresponds to decreasing $\tau$. We have

$$\lim_{\tau\to\infty} f(\tau)\eta_1(\tau) = [\lim_{\tau\to\infty} f(\tau+\tau_{12})/f(\tau)]\,[\lim_{\tau\to\infty} f(\tau)\eta_2(\tau)] \qquad (1.10)$$

provided that these limits exist. Consequently, for all curves except the dividing curve we have

$$\lim_{\tau\to\infty} \eta_1(\tau)\, e^{-c\tau} = e^{-c\tau_{12}} \lim_{\tau\to\infty} \eta_2(\tau)\, e^{-c\tau} \tag{1.11}$$

On the dividing curve we have

$$\lim_{\tau\to\infty} \eta_1(\tau)\, e^{-\tau} = e^{-\tau_{12}} \lim_{\tau\to\infty} \eta_2(\tau)\, e^{-\tau} \tag{1.12}$$

In the case $0 < c < 1$, which is treated in the same manner, the dividing curve approaches asymptotically the straight line $C$ defined by the equation $\vartheta = c\eta$. On this curve we have the finite limit

$$\lim \eta(\tau)\, e^{-c\tau} = c^{-1} \lim \vartheta(\tau)\, e^{-c\tau} \tag{1.13}$$

while on the remaining integral curves we have

$$\lim \eta(\tau)\, e^{-\tau} = \lim \vartheta(\tau)\, e^{-\tau} \tag{1.14}$$

Finally, when $c = 1$, we have the finite limit

$$\lim \eta(\tau)\, e^{-\tau} = \lim \vartheta(\tau)\, e^{-\tau} \tag{1.15}$$

on the dividing curve, and

$$\lim \eta(\tau)\, e^{-\tau}\tau^{-1} = \lim \vartheta(\tau)\, e^{-\tau}\tau^{-1} \tag{1.16}$$

on the remaining curves [5].

In the second subcase when $(c-1)^2 + 4nc > 0$, the singularity is an unstable node (Fig. 1b). On the half-line

$$\vartheta = z_1(\eta - \eta_c), \quad \eta > \eta_c, \quad z_1 = {}^1/_2\,(c+1) + [{}^1/_4\,(c-1)^2 + nc]^{1/2}$$

the inclination of the integral curves is given by

$$\frac{d\vartheta}{d\eta} = c + 1 - \frac{c\eta}{z_1(\eta-\eta_c)} + \frac{\eta^n}{z_1(\eta-\eta_c)} > \tag{1.17}$$

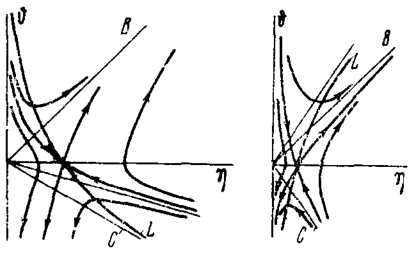
$$> c + 1 - \frac{c\eta}{z_1(\eta-\eta_c)} + \frac{\eta_c^{\,n} + n\eta_c^{n-1}(\eta-\eta_c)}{z_1(\eta-\eta_c)} = z_1$$

For this reason an integral curve appearing to the left of the straight line $\vartheta = z_1(\eta - \eta_c)$ when $\vartheta > 0$, cannot pass to the right of this line and consequently cannot intersect the $O\eta$-axis. Therefore the integral curve can intersect the $O\eta$-axis not more than twice (disregarding the fact that it emerges from a node). Curves which do not intersect the $O\eta$-axis at all exist. Such, in particular, is the dividing curve. Indeed, $\min(c, 1) < z_1$, i.e. the line $\vartheta = z_1(\eta - \eta_c)$ intersects $B$ when $c < 1$ and intersects $C$ when $c > 1$. It follows that if the dividing curve did intersect $O\eta$, it would have to remain above the line $\vartheta = z_1(\eta - \eta_c)$ and could not have $B$ or $C$ as its asymptote. Thus when $n \leqslant -1$ the integral curves have the form depicted on Fig. 1b.

The rate at which the value of the solution increases with $\tau \to \infty$ is the same as in the case $(c-1)^2 + 4nc < 0$.

Finally, the case $(c-1)^2 + 4nc = 0$ differs from the case $(c-1)^2 + 4nc > 0$ only in that all integral curves at the node are tangent to $\vartheta = z_1(\eta - \eta_c)$.

2. Let $c < 0$ and $\gamma = -1$. The singularity of the system (1.3) is $\eta = \eta_c$, $\vartheta = 0$, $\eta_c^{n-1} = -c$. Linearizing (1.3) just as it was done in deriving (1.5) we again obtain (1.5), but since $c < 0$, the singularity is now a saddle-type singularity.




(a) (b)

Fig. 2.

Let $c > -1$. Having assigned the directions to the integral axes (indicated by arrows on Fig. 2a) we conclude that when $n \leqslant -1$, curves exist which approach $O\vartheta$ asymptotically.

It is evident that if $\eta \to 0$ and $\vartheta \to -\infty$ with increasing $\tau$, then $\tau < \tau_{max}$ ($\vartheta$ at infinity is reached in a finite time). Indeed, if $\tau \to \infty$ as $\eta \to 0$, then

$$\eta(\tau) = \eta_0 + \int_0^\tau \vartheta(\xi)\,d\xi, \quad \lim_{\tau\to\infty} \eta(\tau) = 0, \quad \int_0^\infty \vartheta(\xi)\,d\xi = -\eta_0 \tag{1.18}$$

which is impossible as the last integral in the expression diverges. Similarly, if $\eta \to 0$ and $\vartheta \to \infty$ with decreasing $\tau$, then $\tau$ is bounded from below.

As before, from (1.4) we find that $\lim \eta(\tau)e^{-\tau}$ and $\lim \vartheta(\tau)\,e^{-\tau}$ are finite on those segments of the curves on which $\eta$ and $\vartheta \to \infty$ when $\tau \to \infty$, while $\lim\ \eta(\tau)\,e^{-c\tau}$ and $\lim\ \vartheta(\tau)e^{-c\tau}$ are finite on those segments on which $\eta \to \infty$ and $\vartheta \to -\infty$ when $\tau \to -\infty$. If $\eta(\tau) \to \infty$ when $\tau \to \pm\infty$, then we have

$$\lim_{\tau\to\infty} \int_0^\tau e^{c(\tau-\xi)}\eta^n(\xi)\,d\xi = 0, \ \lim_{\tau\to-\infty} \int_0^\tau e^{\tau-\xi}\eta^n(\xi)\,d\xi = 0 \tag{1.19}$$

Therefore on those segments we have $\vartheta - \eta \to 0$ and $\vartheta - c\eta \to 0$ (Fig. 2a).

If $c < -1$, then $L$ and the integral curves are transformed in accordance with Fig. 2b. The asymptotic behavior of the solutions is the same as in the case when $-1 < c < 0$. When $c = -1$ then $L$ degenerates into a straight line $\eta = \eta_c$, while (1.3) and the initial Emden-Fowler system become integrable in quadratures. The behavior of the solutions remains unchanged.

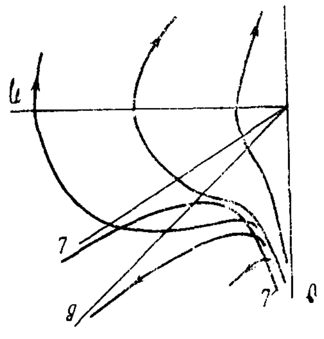


Fig. 3.

3. When $c > 0$ and $\gamma = -1$, the system (1.3) has no singularities, $L$ does not intersect $O\eta$ and reaches its minimum value when $\eta = \eta_m$ and $\eta_m^{n-1} = -c\,/\,n$

Obviously, integral curves which do not intersect $L$ at all or intersect it twice, exist (Fig. 3); when $n \leqslant -1$ the line $\vartheta$ serves as a "double-sided" asymptote for these curves. We shall show that curves intersecting $L$ only once exist. Let us obtain $d\vartheta/d\eta$ for $\vartheta = {}^1\!/_2\,(c+1)\,\eta$. We find that $d\vartheta\,/\,d\eta > {}^1\!/_2\,(c+1)$ when $\eta > \eta_l$ and $4\eta_l^{n-1} = (c-1)^2$, i.e. an integral curve found in the region $\eta > \eta_l$, $\vartheta > {}^1\!/_2\,(c+1)\,\eta$, cannot leave it and must go to infinity.

Let $c > 1$. From (1.4) we find

$$\vartheta < (c-1)^{-1}\,[(\vartheta_0 - \eta_0)\,e^{c\tau} + (c\eta_0 - \vartheta_0)\,e^{\tau}] \tag{1.20}$$

i.e. a curve intersecting $B$ will also intersect $L$. Therefore the integral curves can go to infinity ($\eta$, $\vartheta \to +\infty$) only if they lie above $B$. Amongst these curves we find a unique (dividing) curve approaching $B$ asymptotically.

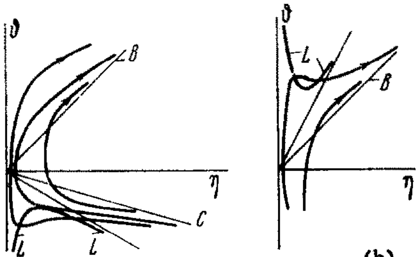
On the dividing curve we have the finite limit $\lim\ \eta(\tau)\ e^{-\tau}$ with $\tau \to \infty$ etc., similarly to the case $c > 0$, $\gamma = 1$. When $c < 1$ the dividing curve approaches $C$ etc. again similarly to the case $c > 0$, $\gamma = 1$.

4. When $c < 0$ and $\gamma = 1$, we arrive at the patterns shown on Fig. 4a ($c > -1$) or Fig. 4b ($c < -1$). In this case the limits

$$\lim_{\tau\to\infty} \eta(\tau) e^{-\tau} = \lim_{\tau\to\infty} \vartheta(\tau) e^{-\tau}, \quad \lim_{\tau\to-\infty} \eta(\tau) e^{-c\tau} = \frac{1}{c} \lim_{\tau\to-\infty} \vartheta(\tau) e^{-c\tau}$$

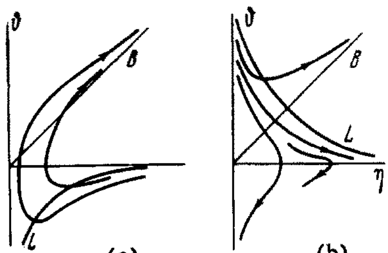
are always finite, and this also holds for the case $c = -1$, when (1.3) and the initial Emden-Fowler equations are integrable in quadratures.

5. When $c = 0$ we have Fig. 5a ($\gamma = 1$) and 5b ($\gamma = -1$). The properties of the



(a) (b)

Fig. 4.



(a) (b)

Fig. 5.

solutions approaching asymptotically $O\vartheta$ or $B$, match those of other similar cases. Let us consider the curves approaching asymptotically $O\eta$. When $\gamma = -1$ we have only one such curve, a dn in place of (1.4) we have

$$\vartheta = \vartheta_0 e^{\tau} - \int_0^{\tau} e^{\tau-\xi} \eta^n(\xi)\, d\xi$$

$$\eta = \eta_0 - \vartheta_0 + \vartheta_0 e^{\tau} - \int_0^{\tau} (e^{\tau-\xi} - 1)\, \eta^n(\xi)\, d\xi \tag{1.21}$$

From (1.21) and the condition that $\vartheta \to 0$ as $\tau \to \infty$ we find that for large $\tau$, $\eta(\tau)$ increases as the integral of $\eta^n(\tau)$, i.e. $\eta \sim \tau^{\nu}$, $\nu = (1-n)^{-1}$. In the case shown on Fig. 5a, the behavior of $\eta$ as $\tau \to -\infty$ is identical.

6. When $2 + \sigma - \alpha = 0$ we must set in (1.2) $\delta = 0$. Assuming in addition that $\alpha\beta - \beta = 1$, we arrive at the system (1.3) of the same form as in the previous case ($c = 0$), but with a different relation connecting $\eta$ and $w$.

**2. Boundary value problems bounded at zero and at infinity.** Knowing the behavior of $\eta(\tau)$ over the range $-\infty < \tau < \infty$ we can establish the properties of $w(\rho)$ for $0 < \rho < \infty$ and find out, in particular, how $w$ changes when $\rho \to 0$ and $\rho \to \infty$. This will enable us to estimate the number and certain properties of solutions of the boundary value problems with the following conditions holding: $w(R) = 1$ and $w(0)$ or $w(\infty)$ is bounded.

The first problem is especially interesting for its applications [3 - 5]. Certain problems containing initially a large number of parameters [4 and 5] can be reduced to the above problems containing a single new parameter $R^*$ by a change of variables.

Let $\beta > 0$. Then $\rho \to 0$ corresponds to $\tau \to \infty$, and $\rho \to \infty$ corresponds to $\tau \to -\infty$. Let us set in (1.2)

$$a = R, \qquad b = (\beta^{-2} R^{\alpha-\sigma-2})^{1/(n-1)}$$

Then the boundary condition $w(R) = 1$ yields $\eta(0) = b^{-1}$. Let us consider a problem in which $w(0)$ is bounded. This is equivalent to the requirement that a finite value of $\lim \eta(\tau)e^{-\tau}$ exists when $\tau \to \infty$.

The most interesting result is obtained when $c \geqslant 1$ and $\gamma = 1$. In this case $\eta(\tau)e^{-\tau}$ is bounded provided that a parametric equation of the dividing curve can be written in terms of $\eta(\tau)$ and $\vartheta(\tau)$. The latter condition together with the condition $\eta_0 = b^{-1}$ defines the required solutions of the boundary value problem. For a given value of $R$ the number of solutions corresponds to the number of times the dividing curve intersects the line $\eta = \eta_* = \text{const} = b^{-1}(R)$. Let $(c-1)^2 + 4nc < 0$. We denote by $\eta_{*1}$, $\eta_{*2}$, ... the values of $\eta$ at the points of intersection of the dividing curve with $O\eta$, their numbering corresponding to the motion along the curve in the direction of decreasing $\tau$. When $\eta_* < \eta_{*1}$ the boundary value problem has no solution, when $\eta_* > \eta_{*2}$ it has one solution, when $\eta_{*1} < \eta_0 < \eta_{*3}$ it has two solutions, etc. Intervals of variation of $\eta_*$ (or $R$) exist on which $k$ solutions exist ($k$ denoting any positive number). When $\eta_0 = \eta_c$ the problem has an enumerable set of solutions. Since

$$\frac{dw}{d\rho} = b\beta^{-1}R^{-1}e^{(\beta-1)\tau}(\eta - \vartheta) > 0$$

$w$ increases from some value $w_m$ to unity when $\rho$ increases from zero to $R$, and $w_m = b \lim \eta e^{-\tau}$, $\tau \to \infty$. Let us investigate the behavior of $w_m(R)$. When $R \to 0$ then $b^{-1} = \eta_* \to \infty$ and (1.9) gives

$$\lim_{\eta_0\to\infty} [\eta_0^{-1} \lim_{\tau\to\infty} \eta(\tau, \eta_0) e^{-\tau}] =$$

$$= 1 - \lim_{\eta_0\to\infty} \frac{1}{\eta_0} \int_0^\infty e^{-\xi} [\eta(\xi, \eta_0) - \vartheta(\xi, \eta_0)]\, d\xi =$$

$$= 1 - \lim_{\tau_{12}\to\infty} \frac{1}{\eta_0} \int_0^\infty e^{-\xi} [\eta(\xi + \tau_{12}, \eta_0^*) - \vartheta(\xi + \tau_{12}, \eta_0^*)]\, d\xi =$$

$$= 1 - \lim_{\tau_{12}\to\infty} \frac{e^{\tau_{12}}}{\eta_0} \int_{\tau_{12}}^\infty e^{-\xi} [\eta(\xi, \eta_0^*) - \vartheta(\xi, \eta_0^*)]\, d\xi = 1 \tag{2.1}$$

Here $\eta(\tau, \eta_0)$ containing the variable $\eta_*$ is expressed in terms of some $\eta(\tau, \eta_0^*)$ containing a fixed initial value $\eta_0^*$ by

$$\eta(\tau, \eta_0) = \eta(\tau + \tau_{12}(\eta_0), \eta_0^*)$$

Proceeding as before from (1.9), we obtain the derivative

$$\frac{dw_m}{dR} = \frac{d\eta_0}{dR}\frac{d}{d\eta_0}\left[-\frac{1}{\eta_0}\int_0^\infty e^{-\xi}[\eta(\xi, \eta_0) - \vartheta(\xi, \eta_0)]\, d\xi\right] =$$

$$= \frac{d\eta_0}{dR}\left[\frac{1}{\eta_0}(1 - w_m) - \frac{1}{\eta_0}\int_0^\infty \frac{1}{\vartheta_0}\left[\vartheta(\xi, \eta_0) - \frac{d}{d\xi}\vartheta(\xi, \eta_0)\right] e^{-\xi} d\xi\right] =$$

$$= -\frac{1}{\beta R}\,\frac{\eta_0 - \vartheta_0}{\vartheta_0}\, w_m \tag{2.2}$$

by using the relations of the type

$$\frac{d\eta(\xi,\eta_0)}{d\eta_0}=\frac{1}{\vartheta_0}\frac{d\eta(\xi,\eta_0)}{d\xi}=\frac{1}{\vartheta_0}\vartheta(\xi,\eta_0) \tag{2.3}$$

and integrating the terms containing $d\vartheta / d\xi$ by parts.

Consider a set of solutions such that the point $(\eta_0, \vartheta_0)$ lies on the segment of the dividing curve emerging from the point $(\eta_{*1}, 0)$ and going to infinity. For these solutions we have $dw_m/dR<0$, i.e. $w_m$ decreases with increasing $R$ for $0<R<R_{*1}$. When $R_{*2}<R<R_{*1}$ solutions exist such that the point $(\eta_0, \vartheta_0)$ lies on the curl of

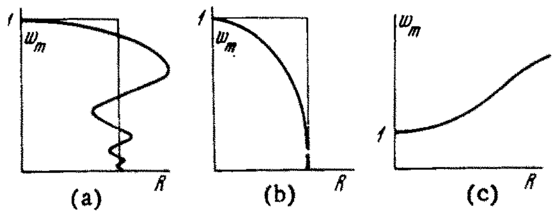



Fig. 6.

the dividing curve connecting $\eta_{*1}$ and $\eta_{*2}$. In these solutions $w_m$ increases with increasing $R$, etc. As the result we obtain the relation $w_m(R)$ in accord with Fig. 6a. Analogous relations for $c\geqslant 1$, $\gamma\geqslant 1$ and $(c-1)^2+4nc>0$ (Fig. 6b) and for $c\geqslant 1$, $\gamma=-1$ (Fig. 6c) are obtained in a similar manner. In the latter case a solution of the boundary value problem exists and is unique for all $R$.

When $c<1$ a two-parameter family of functions $\eta(\tau)$ such that $\lim \eta(\tau)e^{-\tau}$ is finite as $\tau\to\infty$, exists in all cases. Therefore the boundary value problem in question has a one-parameter family of solutions and this indicates that the corresponding physical problem is "underdefined".

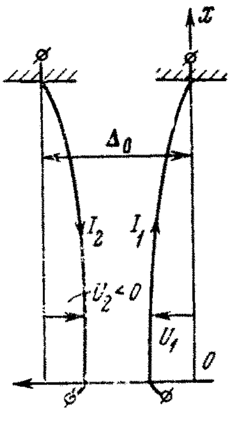



Fig. 7.

Similarly, the boundary value problem with the condition that " $w$ is bounded when $\rho\to\infty$" either has no solutions, or has a family of solutions. When $\beta<0$, the statements referring to the problem with the condition of boundedness at zero can, after obvious changes, be applied to the problem with the condition of boundedness at infinity and vice versa.

Let us give an example leading to a boundary value problem for the Emden-Fowler equation with $n=-1$ Consider the planar forms of equilibrium of two vertical heavy filaments situated close to each other, with a current flowing through them (Fig. 7). Expressing the force per unit length of the filaments by the displacements $u_1(x)$ and $u_2(x)$, we obtain, in accordance with [5], $u_2\equiv -u_1$. Setting now

$$(\Delta_0-2u_1)/\Delta_0=w,\quad \rho=\lambda x,\quad \lambda=\mu_0 I_1 I_2/\pi\rho_1 g\Delta_0^2 \tag{2.4}$$

where $\rho_1 g$ is the weight per unit length of the filament, we arrive at the following expression in $w$

$$\frac{d}{d\rho}\left(\rho\frac{dw}{d\rho}\right)-\gamma w^{-1}=0 \tag{2.5}$$

Substitution $\rho = {}^1/_4\, r^2$ leads to the Emden-Fowler equation with a different value of $\sigma$

$$\frac{d}{dr}\left(r\,\frac{dw}{dr}\right) - \gamma r w^{-1} = 0 \qquad (2.8)$$

The boundary conditions now are: $w(0)$ bounded and $w(\lambda l) = 1$ if (2.5) is used, and $w(2\sqrt{\lambda l}) = 1$ in the case (2.6). Dependence of the solutions on the parameter corresponds to Fig. 6a ($\gamma = 1$, attraction), or to Fig. 6c ($\gamma = -1$, repulsion). In the latter case $w_m = w(0)$ is the maximum value of $w$. These curves can be used in assessing the stability of the equilibrium [5]. In addition, in the repulsion case nonplanar forms of equilibrium and formal solutions describing forms with an apex analogous to those obtained in [5], are also possible. If the dividing curve corresponds to the "usual" forms, then the forms with an apex will have a corresponding combination of the dividing curve with a curve on which $\vartheta \to -\infty$ as $\gamma \to 0$ (Fig. 3).

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Translated by L. K.

# ON THE MAXIMUM VALUE OF THE RADIUS OF THE CONTACT AREA BETWEEN A PUNCH AND A LAYER

PMM Vol. 35, №6, 1971, pp. 1047-1052
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(Received February 6, 1971)

The solution of the following problem of the elasticity theory is given for an infinite weightless homogeneous isotropic layer: a normal concentrated force acts at one of the boundaries of the layer, pressing it against a rigid smooth punch, represented by a convex body of revolution whose axis coincides with the support line of the concentrated force; one has to determine the largest possible value of the radius of the contact area between the punch and the layer for different punches and for different magnitudes of the concentrated force.

**1.** We consider the layer in a cylindrical system of coordinates $rOz$, with origin at the point of application of the concentrated normal force $Q$. The $z$-axis is pointed